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New q -deformed coherent states with an explicitly known resolution of unity

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Abstract

We construct a new family of q -deformed coherent states $|z\rangle_q$, where $0 < q < 1$. These states are normalizable on the whole complex plane and continuous in their label z . They allow the resolution of unity in the form of an ordinary integral with a positive weight function obtained through the analytic solution of the associated Stieltjes power-moment problem and expressed in terms of one of the two Jackson's q -exponentials. They also permit exact evaluation of matrix elements of physically-relevant operators. We use this to show that the photon number statistics for the states is sub-Poissonian and that they exhibit quadrature squeezing as well as an enhanced signal-to-quantum noise ratio over the conventional coherent state value. Finally, we establish that they are the eigenstates of some deformed boson annihilation operator and study some of their characteristics in deformed quantum optics.

PACS numbers: 02.30.-f, 03.65.Fd, 42.50.Ar, 02.20.-a

1. Introduction

The conventional coherent states (CS) [1–3], dating back to the birth of quantum mechanics [4], are specific superpositions $|z\rangle$, parametrized by a single complex number z , of the eigenstates $|n\rangle$ of the harmonic oscillator number operator $N = a^\dagger a$. They have properties similar to those of the classical radiation field and may be constructed in three equivalent ways: (i) by defining them as eigenstates of the annihilation operator a , (ii) by applying a unitary displacement operator on the vacuum state $|0\rangle$ (such that $a|0\rangle = 0$) and (iii) by considering them as quantum states with a minimum uncertainty relationship.

In recent years, other classes of states often termed ‘nonclassical’ have attracted a lot of attention in quantum optics, as well as in other areas ranging from solid state physics to cosmology. Such states exhibit some purely quantum-mechanical properties, such as squeezing, higher-order squeezing, antibunching, sub-Poissonian statistics, etc (for a review

see [5] and references quoted therein). Among them, one finds the so-called generalized coherent states (GCS), arising in connection with algebras other than the oscillator one [6]. They may be defined by extending one of the three equivalent definitions of the conventional CS, which become inequivalent for more complicated algebras. One may therefore distinguish between annihilation-operator [7], displacement-operator [8] and minimum-uncertainty or intelligent GCS [9].

From a mathematical viewpoint, it has been noted [2] that to be acceptable GCS should satisfy a minimum set of conditions: (i) normalizability (as any vector of a Hilbert space), (ii) continuity in the label z and (iii) existence of a resolution of unity with a positive-definite weight function (implying that the states form an (over) complete set). If the first two conditions are rather easy to fulfil, the third one, on the contrary, imposes some severe restrictions on possible GCS. Determining the existence of a unity resolution relation is indeed a difficult task, which has not been solved for many of the putative GCS known in the literature. Recent progress in this domain has been achieved by using some properties of the inverse Mellin transform [10, 11].

A class of GCS, which has evoked a lot of interest, is connected with deformations of the canonical commutation relations or, equivalently, with deformed oscillator algebras. Since the introduction of q -deformed CS [12–14] associated with the so-called *maths*-type [12, 15] and *physics*-type [14, 16] q -deformed bosons, various generalizations have been considered (for a review see [5]) including q -deformed CS associated with two-parameter deformations [17]. This has culminated in the definition of ‘nonlinear’ CS [18], for which the deformation is governed by some function of N .

Few q -deformed CS have been endowed with a unity resolution relation. For those connected with *maths*-type and *physics*-type q -deformed bosons, the latter has been written as a q -integral [19] with a weight function expressed in terms of some q -exponential [12, 20]. An alternative formulation has been proposed in terms of an ordinary integral, but the corresponding weight function is then only known through the inverse Fourier transform of some given function [21]. Another type of q -deformed CS has also been shown to admit a unity resolution relation in the form of an ordinary integral with a weight function expressed as a Laplace transform of some given function [22].

The purpose of the present paper is to prove that in contrast with known q -deformed CS, it is possible to build some new ones admitting a unity resolution relation expressed in terms of an ordinary integral with an explicitly known positive weight function. Moreover, such GCS will be shown to possess some interesting nonclassical properties.

In section 2, new q -deformed physical states are introduced and shown to qualify as GCS in the sense described above. Their geometrical and physical properties in quantum optics are studied in section 3. In section 4, their description in terms of q -deformed bosons is considered. Finally, section 5 contains the conclusion.

2. New q -deformed coherent states in terms of boson operators

Let us consider a new family of harmonic oscillator physical states

$$|z\rangle_q = \mathcal{N}_q^{-1/2}(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]_q!}} |n\rangle \quad (2.1)$$

labelled by $q \in (0, 1)$ and $z \in \mathbb{C}$. Here $|n\rangle = (n!)^{-1/2} (a^\dagger)^n |0\rangle$ is an n -boson state and

$$\begin{aligned} [n]_q! &\equiv [n]_q [n-1]_q \dots [1]_q && \text{if } n = 1, 2, \dots \\ &\equiv 1 && \text{if } n = 0 \end{aligned} \quad (2.2)$$

where

$$[n]_q \equiv \frac{1 - q^{-n}}{q - 1} = q^{-n} \{n\}_q \tag{2.3}$$

differs from the corresponding quantity $\{n\}_q \equiv (1 - q^n)/(1 - q)$ for maths-type q -deformed bosons [12, 13, 15] by the n - and q -dependent factor q^{-n} . Note that the q -factorial (2.2) can also be written in terms of the q -gamma function $\Gamma_q(x)$ [23] as

$$[n]_q! = q^{-n(n+1)/2} \Gamma_q(n + 1). \tag{2.4}$$

For $q \rightarrow 1^-$, $[n]_q$, $[n]_q!$ and $\Gamma_q(n)$ go to n , $n!$ and $\Gamma(n)$, respectively.

Let us now show that the states $|z\rangle_q$ are GCS in the sense described in section 1.

2.1. Normalizability

The normalization condition of (2.1), ${}_q\langle z|z\rangle_q = 1$, leads to

$$\mathcal{N}_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{\{n\}_q!} (qx)^n = E_q[(1 - q)qx] \tag{2.5}$$

where $x \equiv |z|^2$, $\{n\}_q!$ is defined in a way similar to $[n]_q!$, and $E_q(z) \equiv \prod_{k=0}^{\infty} (1 + q^k z)$ is one of the two q -exponentials introduced by Jackson (such that $\lim_{q \rightarrow 1^-} E_q[(1 - q)z] = \exp(z)$) [19]. Since $E_q(z)$ converges absolutely for $0 < |q| < 1$ and any z [23], $\mathcal{N}_q(x)$ is an entire function. Moreover, it is positive on the positive real axis. The states (2.1) are therefore normalizable on the whole complex plane. For $q \rightarrow 1^-$, they reduce to the conventional CS [1–3],

$$|z\rangle = \mathcal{N}^{-1/2}(x) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \quad \mathcal{N}(x) = e^x. \tag{2.6}$$

A convenient way to visualize how $|z\rangle_q \xrightarrow{q \rightarrow 1} |z\rangle$ is to display the function

$$d_q(n) = \frac{[n]_q!}{n!} \tag{2.7}$$

for different values of the parameter q . This is done in figure 1. We note that $d_q(n)$ is always greater than 1. For a given q it is an increasing function of n , while for a given n its departure from 1 increases for decreasing q values. It follows that for such q values the convergence of the series defining $\mathcal{N}_q(x)$ in (2.5) becomes increasingly fast as compared with that defining the exponential. This is confirmed in figure 2, where $\mathcal{N}_q(x)$ is displayed for several q values and compared with $\mathcal{N}(x)$. Observe that for $x \ll 1$, $\mathcal{N}_q(x)$ behaves as

$$\mathcal{N}_q(x) \simeq 1 + qx + O(x^2). \tag{2.8}$$

2.2. Continuity in z

The states $|z\rangle_q$, defined in (2.1), are continuous in z if

$$|z - z'| \rightarrow 0 \quad \Rightarrow \quad \||z\rangle_q - |z'\rangle_q\|^2 \rightarrow 0. \tag{2.9}$$

Since

$$\||z\rangle_q - |z'\rangle_q\|^2 = 2(1 - \text{Re}_q \langle z|z'\rangle_q) \tag{2.10}$$

where

$${}_q\langle z|z'\rangle_q = [\mathcal{N}_q(|z|^2)\mathcal{N}_q(|z'|^2)]^{-1/2} E_q[(1 - q)qz^*z] \tag{2.11}$$

is a continuous function, it follows that condition (2.9) is satisfied.

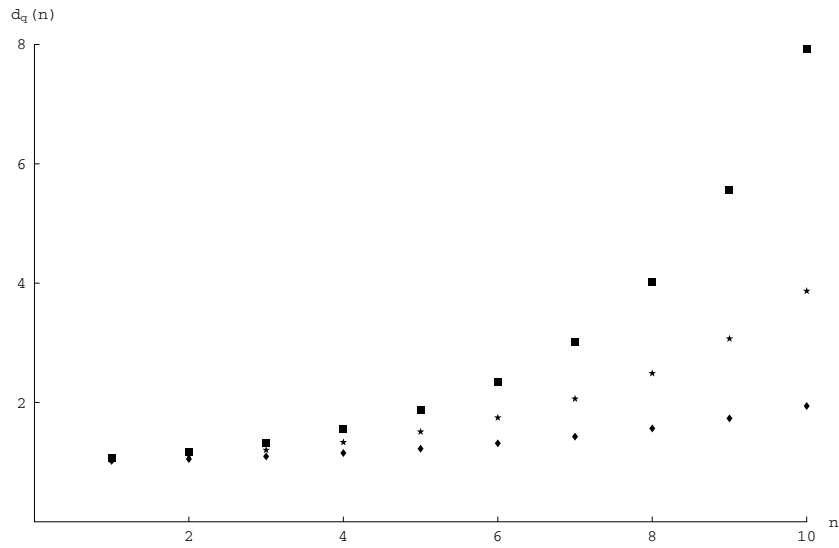


Figure 1. The ratio $d_q(n)$ of equation (2.7) versus n for $q = 0.98$ (diamonds), $q = 0.96$ (stars) and $q = 0.94$ (squares).

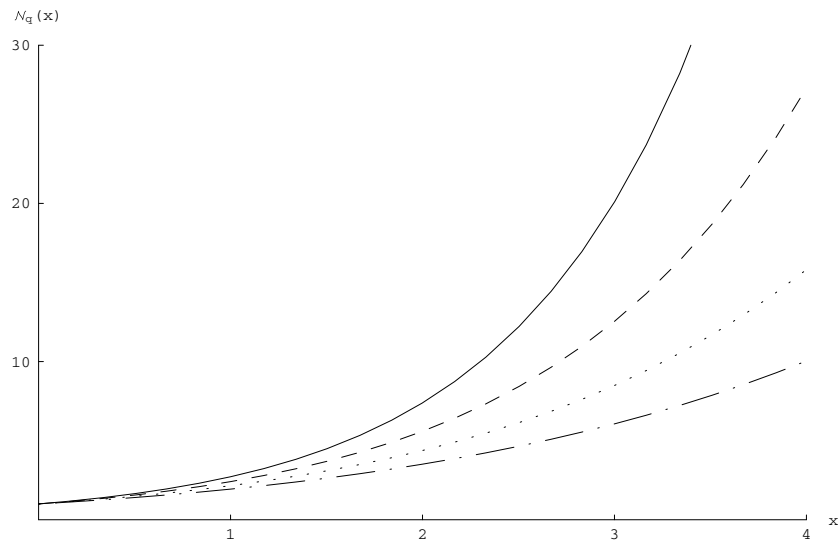


Figure 2. The normalization function $N_q(x)$ of equation (2.5) versus x for $q = 1$ (solid line), $q = 0.9$ (dashed line), $q = 0.8$ (dotted line) and $q = 0.7$ (dot-dashed line).

2.3. Resolution of unity

The states $|z\rangle_q$ give rise to a resolution of unity with weight function $W_q(x)$ if

$$\int \int_{\mathbb{C}} d^2z |z\rangle_q W_q(|z|^2)_q \langle z| = \sum_{n=0}^{\infty} |n\rangle \langle n| = I. \tag{2.12}$$

On using expansions of $|z\rangle_q$ and ${}_q\langle z|$ in terms of n -boson kets and bras, such a condition is transformed into

$$\sum_{n=0}^{\infty} \left(\frac{\pi}{[n]_q!} \int_0^{\infty} dx \frac{x^n W_q(x)}{\mathcal{N}_q(x)} \right) |n\rangle\langle n| = I. \tag{2.13}$$

It can be satisfied provided the function

$$\tilde{W}_q(x) \equiv \pi \frac{W_q(x)}{\mathcal{N}_q(x)} \tag{2.14}$$

is a solution of the following Stieltjes power-moment problem:

$$\int_0^{\infty} dx x^n \tilde{W}_q(x) = [n]_q! \quad n = 0, 1, 2, \dots \tag{2.15}$$

From relation (2.4) between the q -factorial and the q -gamma function and from a q -analogue of the Euler gamma integral [24], we directly obtain a solution of (2.15) in the form

$$\tilde{W}_q(x) = \frac{1-q}{\ln q^{-1}} \{E_q[(1-q)x]\}^{-1} \tag{2.16}$$

from which the relation

$$W_q(x) = \frac{1-q}{\pi \ln q^{-1}} \frac{E_q[(1-q)qx]}{E_q[(1-q)x]} \tag{2.17}$$

follows.

It is easy to check that in the limit $q \rightarrow 1^-$, $\tilde{W}_q(x) \rightarrow \tilde{W}(x) = e^{-x}$ and $W_q(x) \rightarrow W(x) = \pi^{-1}$, thus giving back the corresponding results for the conventional CS (2.6) [1–3]. For $x \ll 1$, we get

$$\tilde{W}_q(x) \simeq \frac{1-q}{\ln q^{-1}} [1 - x + O(x^2)] \quad W_q(x) \simeq \frac{1-q}{\pi \ln q^{-1}} [1 - (1-q)x + O(x^2)]. \tag{2.18}$$

In figures 3 and 4, the weight functions $\tilde{W}_q(x)$ and $W_q(x)$ are displayed for several q values and compared with $\tilde{W}(x)$ and $W(x)$, respectively.

We therefore conclude that the new physical states $|z\rangle_q$, defined in (2.1), qualify as GCS in the sense described in section 1.

It is worth emphasizing, however, that in contradistinction to the case of the conventional CS, the solution $\tilde{W}_q(x)$ of the power-moment problem (2.15) may not be unique as the so-called Carleman criterion [25] indicates. According to the latter, if a solution exists and $S \equiv \sum_{n=1}^{\infty} a_n$, where $a_n \equiv ([n]_q!)^{-1/(2n)}$, diverges (resp. converges), then the solution is unique (resp. nonunique solutions may exist). The logarithmic test [26], $\lim_{n \rightarrow \infty} (\ln a_n / \ln n) = \lim_{n \rightarrow \infty} (4 \ln n)^{-1} (n+1) \ln q < -1$, actually shows the convergence of S , hence the possible existence of other solutions.

3. Geometrical and physical properties in quantum optics

In this section, we shall proceed to study some geometrical and physical properties of our new q -deformed CS $|z\rangle_q$. For such a purpose, we shall need to evaluate the expectation values of some Hermitian monomials in the boson creation and annihilation operators a^\dagger, a . These are expressible through the derivatives of $\mathcal{N}_q(x)$ as [10, 22]

$${}_q\langle (a^\dagger)^r a^r \rangle_q = \frac{x^r}{\mathcal{N}_q(x)} \frac{d^r \mathcal{N}_q(x)}{dx^r} \quad r = 0, 1, 2, \dots \tag{3.1}$$

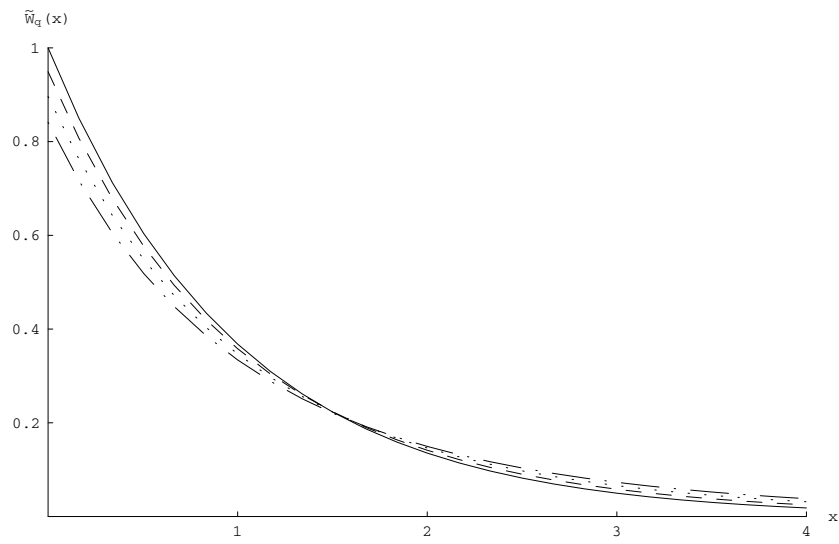


Figure 3. The weight function $\tilde{W}_q(x)$ of equation (2.16) versus x for $q = 1$ (solid line), $q = 0.9$ (dashed line), $q = 0.8$ (dotted line) and $q = 0.7$ (dot-dashed line).

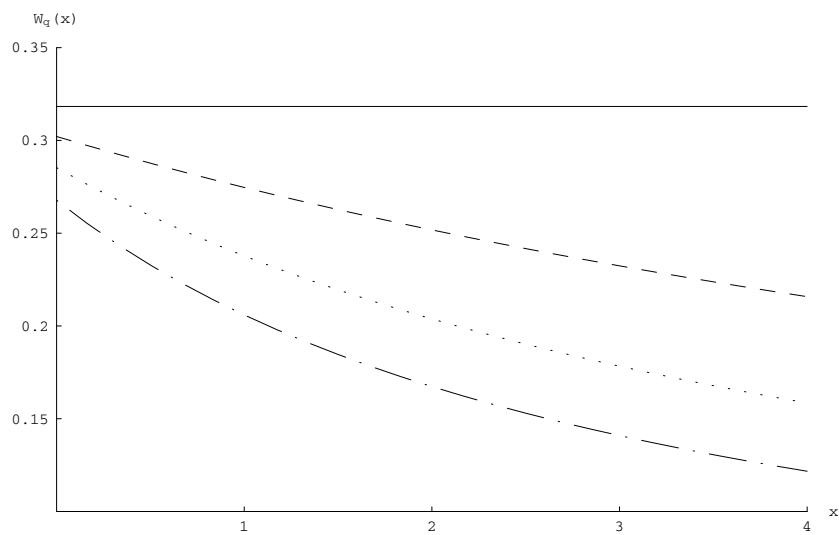


Figure 4. The weight function $W_q(x)$ of equation (2.17) versus x for $q = 1$ (solid line), $q = 0.9$ (dashed line), $q = 0.8$ (dotted line) and $q = 0.7$ (dot-dashed line).

where we use the notation

$${}_q \langle O \rangle_q \equiv {}_q \langle z | O | z \rangle_q. \tag{3.2}$$

More generally, for non-Hermitian monomials, equation (3.1) can be extended to

$${}_q \langle (a^\dagger)^p a^r \rangle_q = (z^*)^p z^r S_q^{(p,r)}(x) \tag{3.3}$$

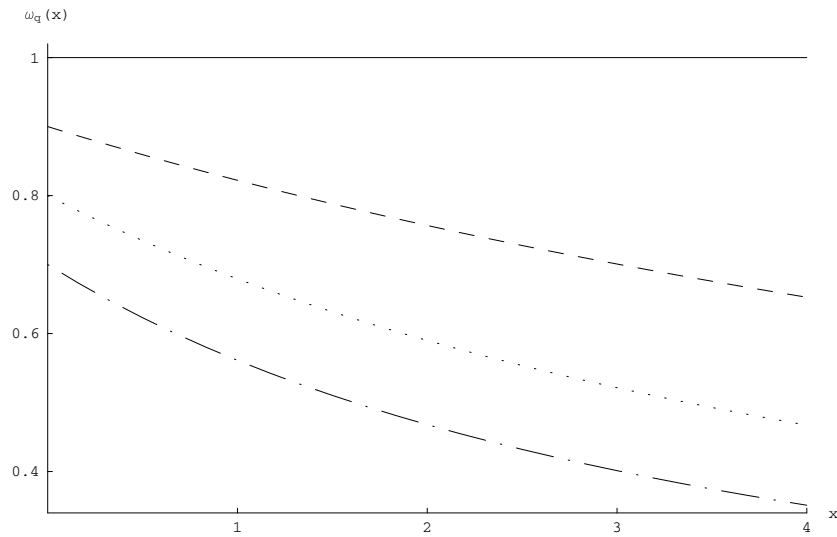


Figure 5. The metric factor $\omega_q(x)$ of equation (3.5) versus x for $q = 1$ (solid line), $q = 0.9$ (dashed line), $q = 0.8$ (dotted line) and $q = 0.7$ (dot-dashed line).

where

$$S_q^{(p,r)}(x) = \frac{1}{\mathcal{N}_q(x)} \sum_{n=0}^{\infty} \left(\frac{(n+p)!(n+r)!}{[n+p]_q![n+r]_q!} \right)^{1/2} \frac{x^n}{n!} \quad r, p = 0, 1, 2, \dots \quad (3.4)$$

3.1. Metric factor

The map from z to $|z\rangle_q$ is a map from \mathbb{C} into a continuous subset of unit vectors in the Hilbert space. It generates in the latter a two-dimensional surface, whose non-flat, circularly symmetric geometry [10] is described in polar coordinates r, θ (i.e. $z = r e^{i\theta}$) by the line element $d\sigma_q^2 = \omega_q(r^2)(dr^2 + r^2 d\theta^2)$, where the metric factor is

$$\omega_q(x) = \frac{d}{dx} \langle N \rangle_q = \left(\frac{x \mathcal{N}'_q(x)}{\mathcal{N}_q(x)} \right)' = \frac{\mathcal{N}'_q(x)}{\mathcal{N}_q(x)} + x \left[\frac{\mathcal{N}''_q(x)}{\mathcal{N}_q(x)} - \left(\frac{\mathcal{N}'_q(x)}{\mathcal{N}_q(x)} \right)^2 \right]. \quad (3.5)$$

Here primes denote the order of differentiations with respect to the variable x . One can interpret $d\sigma_q^2$ as the geometry of the associated classical phase space.

For $x \ll 1$, we obtain

$$\omega_q(x) \simeq q \left[1 - \frac{2q(1-q)}{1+q} x + O(x^2) \right]. \quad (3.6)$$

Figure 5, which shows $\omega_q(x)$ for several q values, confirms that over an extended x range, $\omega_q(x) < \omega(x) = 1$, corresponding to the flat geometry of the conventional CS.

3.2. Photon number distribution

The probability of finding n bosons in the q -deformed CS $|z\rangle_q$ is equal to

$$p_q(n, x) = \frac{x^n}{\mathcal{N}_q(x)[n]_q!}. \quad (3.7)$$

In the limit $q \rightarrow 1^-$, it reduces to a Poisson distribution for the conventional CS.

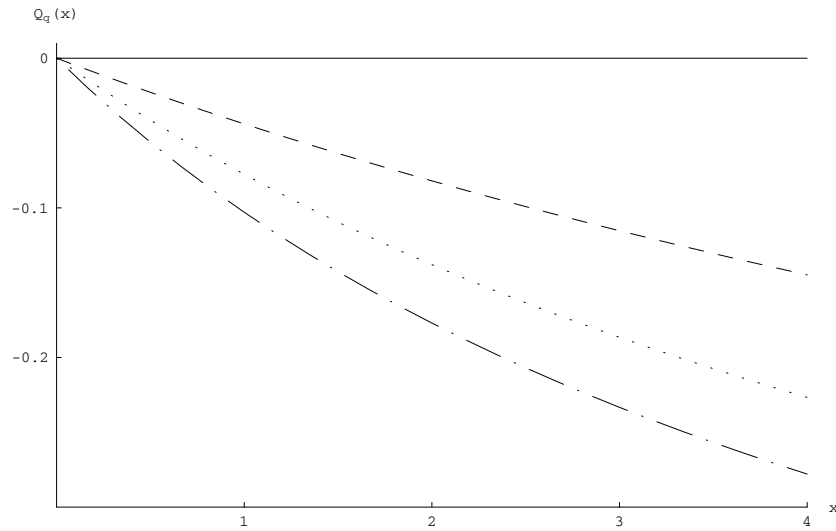


Figure 6. The Mandel parameter $Q_q(x)$ of equation (3.9) versus x for $q = 1$ (solid line), $q = 0.9$ (dashed line), $q = 0.8$ (dotted line) and $q = 0.7$ (dot-dashed line).

Since for the latter the variance of the number operator N is equal to its average, deviations from the Poisson statistics can be measured with the Mandel parameter [27],

$$Q_q(x) = \frac{(\Delta N)_q^2 - \langle N \rangle_q}{\langle N \rangle_q} \quad (\Delta N)_q^2 \equiv \langle N^2 \rangle_q - \langle N \rangle_q^2 \quad (3.8)$$

which vanishes for the Poisson distribution, is positive for a super-Poissonian distribution (bunching effect) and is negative for a sub-Poissonian distribution (antibunching effect). From (3.1), it follows that $Q_q(x)$ can be written as

$$Q_q(x) = x \left(\frac{\mathcal{N}_q''(x)}{\mathcal{N}_q'(x)} - \frac{\mathcal{N}_q'(x)}{\mathcal{N}_q(x)} \right). \quad (3.9)$$

For $x \ll 1$, we obtain

$$Q_q(x) \simeq -\frac{q(1-q)}{1+q}x + O(x^2). \quad (3.10)$$

This result hints at a sub-Poissonian distribution. Numerical study confirms that this is indeed true, as displayed in figure 6.

3.3. Squeezing properties

Let us consider the Hermitian quadrature operators

$$X = \frac{1}{\sqrt{2}}(a + a^\dagger) \quad P = \frac{1}{i\sqrt{2}}(a - a^\dagger). \quad (3.11)$$

Their variances $(\Delta X)^2$ and $(\Delta P)^2$ in any state $|\psi\rangle$ satisfy the conventional uncertainty relation

$$(\Delta X)^2(\Delta P)^2 \geq \frac{1}{4} \quad (3.12)$$

the lower bound being attained by the vacuum state, for which both variances are equal to $\frac{1}{2}$. A state $|\psi\rangle$ is said to be squeezed for the quadrature X (resp. P) if $(\Delta X)^2 < \frac{1}{2}$ (resp. $(\Delta P)^2 < \frac{1}{2}$).

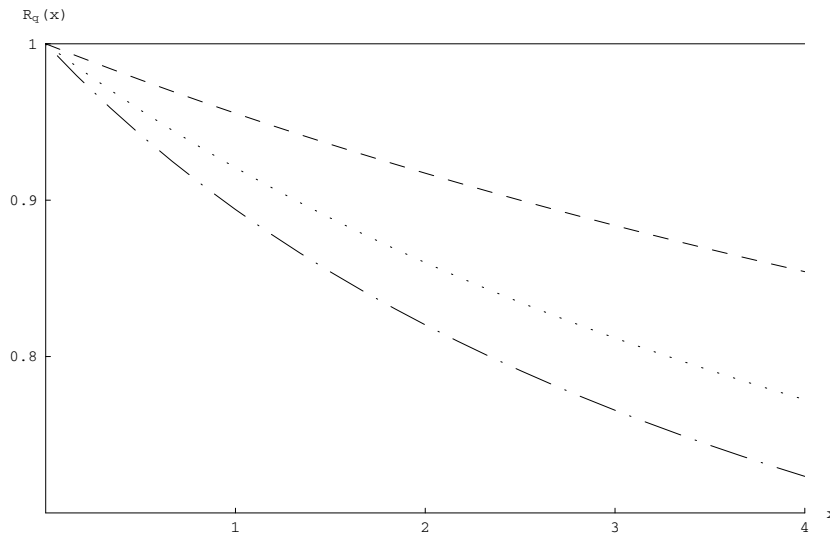


Figure 7. The variance ratio $R_q(x)$ of equation (3.15) versus x for real $z = \sqrt{x}$ and $q = 1$ (solid line), $q = 0.9$ (dashed line), $q = 0.8$ (dotted line) or $q = 0.7$ (dot-dashed line).

For the q -deformed CS $|z\rangle_q$, it is straightforward to show that the variances of X and P are given by

$$(\Delta X)_q^2 = 2(\operatorname{Re} z)^2 \left\{ S_q^{(2,0)}(x) - [S_q^{(1,0)}(x)]^2 \right\} + x [S_q^{(1,1)}(x) - S_q^{(2,0)}(x)] + \frac{1}{2} \quad (3.13)$$

$$(\Delta P)_q^2 = 2(\operatorname{Im} z)^2 \left\{ S_q^{(2,0)}(x) - [S_q^{(1,0)}(x)]^2 \right\} + x [S_q^{(1,1)}(x) - S_q^{(2,0)}(x)] + \frac{1}{2}. \quad (3.14)$$

Since they are related to each other by the transformation $\operatorname{Re} z \leftrightarrow \operatorname{Im} z$, it is enough to study the former.

For small x values, the coefficient of $2(\operatorname{Re} z)^2$ in (3.13) behaves as $q\left(\sqrt{\frac{2q}{1+q}} - 1\right)$, which is negative for $0 < q < 1$. It can be checked numerically that this property holds true over a wide range of x values. Hence, the maximum squeezing in X can be achieved when z is real.

For $0 < z = \sqrt{x} \ll 1$, the ratio

$$R_q(x) = 2(\Delta X)_q^2 \quad (3.15)$$

of the variance $(\Delta X)_q^2$ in $|z\rangle_q$ to the variance $\frac{1}{2}$ in the vacuum state behaves as

$$R_q(x) \simeq 1 + 2q \left(\sqrt{\frac{2q}{1+q}} - 1 \right) x + O(x^2) \quad (3.16)$$

which shows that squeezing is present for small x values. This is confirmed by figure 7, where $R_q(x)$ is plotted against x for $z = \sqrt{x}$ and several q values. We conclude that there is a substantial squeezing increasing with x and with the deformation.

3.4. Signal-to-quantum noise ratio

Finally, let us calculate the signal-to-quantum noise ratio σ_q , which is defined by

$$\sigma_q = \frac{\langle X \rangle_q^2}{(\Delta X)_q^2} \quad (3.17)$$

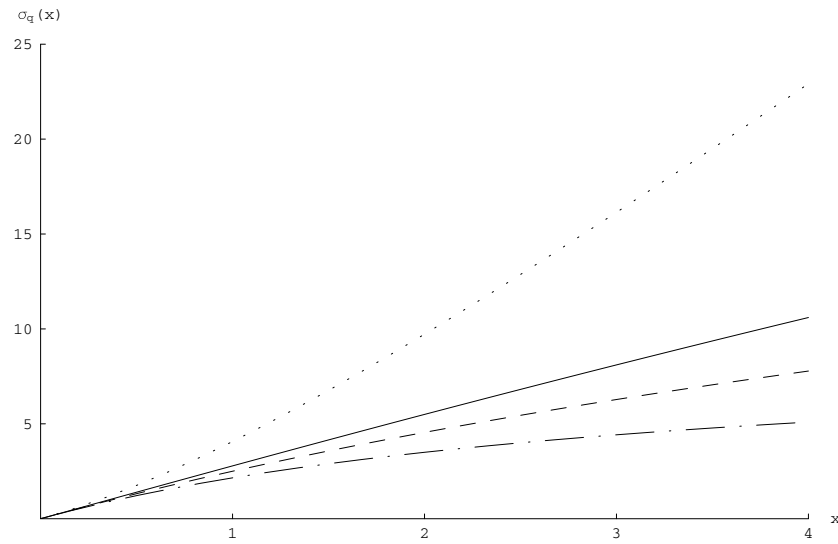


Figure 8. Signal-to-quantum noise ratio plots versus x for $q = 0.7$: σ_q of equation (3.17) for real $z = \sqrt{x}$ (solid line), $4\langle N \rangle_q$ (dashed line), $4\langle N \rangle_q(\langle N \rangle_q + 1)$ (dotted line) and σ_{bq} of equation (4.16) (dot-dashed line).

where $(\Delta X)_q^2$ is given in (3.13) and

$$\langle X \rangle_q^2 = 2(\operatorname{Re} z)^2 [S_q^{(1,0)}(x)]^2. \quad (3.18)$$

It follows that for a given x value, such a ratio is maximum for real z . For $0 < z = \sqrt{x} \ll 1$, it behaves as

$$\sigma_q(x) \simeq 4qx[1 + O(x^2)]. \quad (3.19)$$

For the conventional CS, the ratio σ ($=\sigma_1$) attains the value $4N_s$, where $N_s = \langle N \rangle$ is the number of photons in the signal. In the case of the q -deformed CS $|z\rangle_q$, we may compare $\sigma_q(x)$ to $4\langle N \rangle_q$. For $x \ll 1$, the latter behaves as

$$4\langle N \rangle_q \simeq 4qx \left[1 - \frac{q(1-q)}{1+q}x + O(x^2) \right]. \quad (3.20)$$

On comparing with (3.17), it is clear that there is an improvement on the conventional CS value.

We may also compare $\sigma_q(x)$ to the upper bound $4N_s(N_s + 1) = 4\langle N \rangle_q(\langle N \rangle_q + 1)$ [28], which is attained by a conventional squeezed state [29]. For $x \ll 1$, we obtain

$$4\langle N \rangle_q(\langle N \rangle_q + 1) \simeq 4qx \left[1 + \frac{2q^2}{1+q}x + O(x^2) \right] \quad (3.21)$$

thus confirming the prediction.

As displayed in figure 8, numerical calculations show the validity of the above conclusions over an extended range of x values.

4. Description in terms of q -deformed bosons

4.1. The new q -deformed coherent states in terms of q -deformed bosons

The new q -deformed CS, introduced in section 2 in terms of the conventional boson operators a^\dagger, a , can alternatively be described in terms of some q -deformed boson operators b^\dagger, b , defined in terms of the latter by

$$b^\dagger = \sqrt{\frac{[N]_q}{N}} a^\dagger \quad b = a \sqrt{\frac{[N]_q}{N}} \quad (4.1)$$

where $[N]_q$ is given, as in (2.3), by

$$[N]_q = \frac{1 - q^{-N}}{q - 1}. \quad (4.2)$$

Contrary to the boson operators for which $a^\dagger a = N, a a^\dagger = N + 1$, the q -deformed ones are such that

$$b^\dagger b = [N]_q \quad b b^\dagger = [N + 1]_q. \quad (4.3)$$

From these relations, it follows that they satisfy the commutation relation

$$[b, b^\dagger] = [N + 1]_q - [N]_q = q^{-N-1} \quad (4.4)$$

together with

$$[N, b^\dagger] = b^\dagger \quad [N, b] = -b \quad (4.5)$$

resulting from (4.1). Observe that the N -dependent commutation relation (4.4) may be replaced by an N -independent one provided we substitute a ‘quommutator’ for the ordinary commutator

$$[b, b^\dagger]_{q^{-1}} \equiv b b^\dagger - q^{-1} b^\dagger b = q^{-1}. \quad (4.6)$$

The operators b^\dagger, b act on the same Fock space $\{|n\rangle \mid n = 0, 1, 2, \dots\}$ as a^\dagger, a . From (4.1), we obtain

$$b|n\rangle = \sqrt{[n]_q} |n-1\rangle \quad b^\dagger|n\rangle = \sqrt{[n+1]_q} |n+1\rangle \quad (4.7)$$

with $b|0\rangle = 0$. The n -boson states $|n\rangle$ can, therefore, be rewritten as n -deformed-boson states through the relation

$$|n\rangle = \frac{1}{\sqrt{[n]_q!}} (b^\dagger)^n |0\rangle. \quad (4.8)$$

On inserting equation (4.8) in (2.1) and using the definition of the q -exponential given in (2.5), we can express our new q -deformed CS in terms of the q -deformed boson operators b^\dagger, b as

$$|z\rangle_q = \mathcal{N}_q^{-1/2}(x) E_q[(1-q)qz b^\dagger] |0\rangle. \quad (4.9)$$

From (2.1) and (4.7), it also follows that

$$b|z\rangle_q = z|z\rangle_q \quad (4.10)$$

showing that the q -deformed CS $|z\rangle_q$ are annihilation-operator CS for the q -deformed oscillator algebra generated by N, b^\dagger, b , and defined by equations (4.4) and (4.5). In the limit $q \rightarrow 1^-$, equations (4.9) and (4.10) reduce to known results for the conventional CS, namely, $|z\rangle = \mathcal{N}^{-1/2}(x) \exp(za^\dagger) |0\rangle$ and $a|z\rangle = z|z\rangle$.

Observe that, as announced in section 1, the states (2.1) or (4.9) cannot be obtained by applying a unitary displacement operator on the vacuum if the base field remains the complex

plane. It might, however, prove possible to construct such a displacement operator by using a non-commutative base field, as was done in [30] for the maths-type q -deformed CS.

4.2. Physical properties in deformed quantum optics

The q -deformed boson operators b^\dagger, b may be interpreted as describing ‘dressed’ photons, which may be invoked in phenomenological models explaining some non-intuitive observable phenomena [31]. The physical applications considered in section 3 for ‘real’ photons may therefore be re-examined for those deformed photons.

In such a context, it has been shown [32] that the properties of q -deformed CS, defined as in (4.10), are governed by a characteristic functional

$$\rho_q(n) = \frac{[n+1]_q/[n]_q}{(n+1)/n} \quad (4.11)$$

which takes the value 1 for conventional photons. In the present case, it can easily be proved that $\rho_q(n) > 1$ for $0 < q < 1$.

From this, we deduce the following consequences:

- (i) The Mandel parameter $Q_q(x)$ is negative, which corresponds to a sub-Poissonian distribution. This agrees with the results of section 3.2.
- (ii) The variances of the deformed quadrature operators

$$X_b = \frac{1}{\sqrt{2}}(b + b^\dagger) \quad P_b = \frac{1}{i\sqrt{2}}(b - b^\dagger) \quad (4.12)$$

in any state satisfy the uncertainty relation

$$(\Delta X_b)^2 (\Delta P_b)^2 \geq \frac{1}{4} |\langle [X_b, P_b] \rangle|^2. \quad (4.13)$$

For the q -deformed CS $|z\rangle_q$, such a relation becomes an equality since

$$\begin{aligned} (\Delta X_b)_q^2 &= (\Delta P_b)_q^2 = \frac{1}{2} |\langle [X_b, P_b] \rangle_q| = \frac{1}{2} (\langle [N+1]_q \rangle - \langle [N]_q \rangle) \\ &= \frac{1}{2} \left\{ \frac{1}{\mathcal{N}_q(x)} \sum_{n=0}^{\infty} [n+1]_q \frac{x^n}{[n]_q!} - x \right\}. \end{aligned} \quad (4.14)$$

The states $|z\rangle_q$ are therefore intelligent states for the deformed operators X_b, P_b . However, they are not minimum-uncertainty states as the vacuum uncertainty product provides a global lower bound. Moreover, no squeezing occurs in $|z\rangle_q$ since the ratio

$$R_{bq}(x) = 2q (\Delta X_b)_q^2 \quad (4.15)$$

of $(\Delta X_b)_q^2$ to the variance $1/(2q)$ of X_b in the vacuum state is always greater than 1. This is illustrated in figure 9, where $R_{bq}(x)$ is plotted against x for several q values.

- (iii) The signal-to-quantum noise ratio for deformed photons

$$\sigma_{bq} = \frac{\langle X_b \rangle_q^2}{(\Delta X_b)_q^2} \quad (4.16)$$

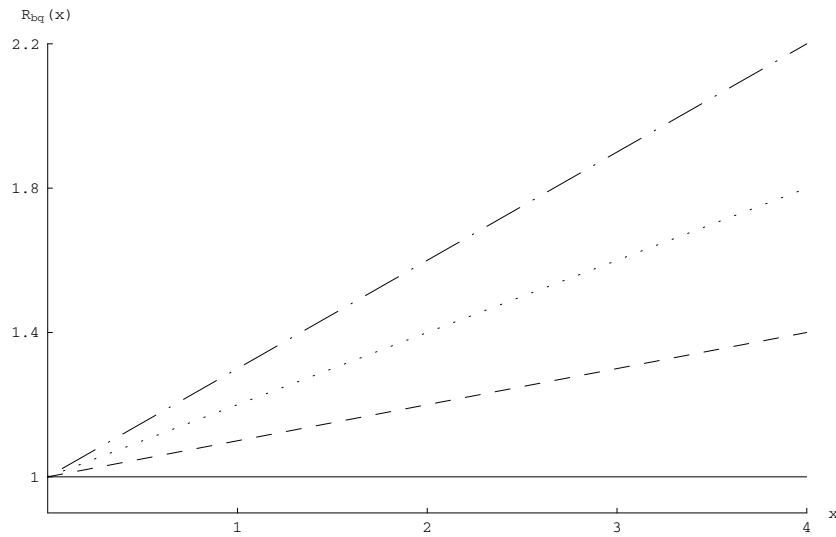


Figure 9. The variance ratio $R_{bq}(x)$ of equation (4.15) versus x for $q = 1$ (solid line), $q = 0.9$ (dashed line), $q = 0.8$ (dotted line) and $q = 0.7$ (dot-dashed line).

is not enhanced over the conventional photon case. Here $(\Delta X_b)_q^2$ is given by (4.14) and $\langle X_b \rangle_q = \sqrt{2} \operatorname{Re} z$; hence, for a given x value, σ_{bq} is maximum for real z . In figure 8, it can be seen that for such values, σ_{bq} is indeed lower than $4\langle N \rangle_q$.

5. Conclusion

In the present paper, we have defined new q -deformed CS by slightly modifying the maths-type q -deformed CS [12, 13]. Such a change has allowed us to give the explicit weight function that satisfies the associated Stieltjes power-moment problem and thus the unity resolution property in the form of an ordinary integral.

The simple form of our CS has also enabled us to provide analytical formulae for calculating matrix elements of physically-relevant operators. With their use, we have investigated some characteristics of our CS relevant to quantum optics. We have established that they exhibit strong nonclassical properties, such as antibunching, quadrature squeezing and enhancement of the signal-to-quantum noise ratio.

Finally, we have shown that this family of states can be expressed as eigenstates of some deformed boson annihilation operator, and we have studied some phenomena in deformed quantum optics.

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